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EXTRAPOLATION PROBLEM FOR FUNCTIONALS OF STATIONARY PROCESSES WITH MISSING OBSERVATIONS

Досліджується задача оптимального лінійного оцінювання функціонала від невідомих значень стаціонарного стохастичного процесу за даними спостережень процесу з шумом. Знайдені формули для обчислення середньоквадратичної похибки та спектральної характеристики оптимальної оцінки функціонала за умови, що спектральні щільності процесів відомі. У випадку, коли вигляд спектральних щільностей невідомий, але задані множини допустимих спектральних щільностей, застосовано мінімаксний метод оцінювання. Для заданих множин допустимих спектральних щільностей визначені найменш сприятливі спектральні щільності та мінімаксні спектральні характеристики оптимальної лінійної оцінки функціонала.

The problem of optimal linear estimation of the functional which depends on unknown values of a stochastic stationary process from observations of the process with noise is considered. Formulas for calculating the mean-square error and the spectral characteristic of the optimal linear estimate of the functional are proposed under the condition of spectral certainty, where the spectral densities of the processes $\xi(t)$ and $\eta(t)$ are exactly known. The minimax (robust) method of estimation is applied in the case where the spectral densities are not known exactly, but sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics are proposed for some special sets of admissible densities.

Effective methods of solution of the problems of interpolation, extrapolation and filtering of stationary sequences and processes were developed by A. N. Kolmogorov [15]. Further analysis can be found in the works by Yu. A. Rozanov [32], E. J. Hannan [10], H. Wold [36,37], T. Nakazi [28], N. Wiener [35], A. M. Yaglom [38,39].

The crucial assumption of most of the methods of estimation of the unobserved values of stochastic processes is that the spectral densities of the considered stochastic processes are exactly known. However, in practice, complete information on the spectral densities is impossible in most cases. In order to solve the problem one can find parametric or nonparametric estimates of the unknown spectral densities. Then, one of the traditional estimation methods is applied. With the help of some examples K. S. Vastola and H. V. Poor [34]. have demonstrated that this procedure can result in significant increasing of the value of error.

In the paper by Ulf Grenander [9], which deals with the problem of extrapolation for stationary processes, a new method of estimation, called minimax, was introduced. The purpose of this method is to search the estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error.

Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H. V. Poor [14]. J. Franke [5], J. Franke and H. V. Poor [6] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities.

The results of research of the problems of the linear optimal estimation of the functionals which depend on the unknown values of stationary sequences and processes are given in papers by M. Moklyachuk [19] – [22]. A minimax technique of estimation for vector-valued stationary stochastic processes was developed by M. Moklyachuk and A. Masyutka in papers [23]–[25]. Solution of estimation problems for periodically correlated stochastic processes were proposed by M. Moklyachuk and I. Dubovetska [4,8]. Estimation problems for functionals which depend on the unknown values of stochastic processes with stationary increments were investigated by M. Luz and M. Moklyachuk [16]–[18]. The problem of interpolation of stationary sequence with missing values was investigated by M. Moklyachuk and M. Sidei [26,27].

Prediction of stationary processes with missing observations was investigated in papers by P. Bondon [1,2], Y. Kasahara, M. Pourahmadi and A. Inoue [13,29], R. Cheng, A. G. Miamee, M. Pourahmadi [3]. The problem of interpolation of stationary sequences was considered in the paper of H. Salehi [33].

In this paper we deal with the problem of the mean-square optimal linear estimation of the functional $A\xi = \int_0^\infty a(t)\xi(t)dt$, which depends on the unknown values of a stochastic stationary process $\xi(t)$ from observations of the process $\xi(t) + \eta(t)$ at time points $t \in \mathbb{R}^-\backslash S$, $S = \bigcup_{l=1}^{s}[-M_l - N_l, \ldots, -M_l]$, $M_l = \sum_{k=0}^{l}(N_k + K_k)$, $N_0 = 0$, $K_0 = 0$. First we consider the case when spectral densities are known, and apply the Hilbert space method to the solution of the estimation problem. We use minimax-robust method of estimation to solve the problem for the given classes of admissible spectral densities in the case when spectral densities are not exactly known.

Classical extrapolation problem of stationary processes

Consider two uncorrelated stochastic processes $\{\xi(t), t \in \mathbb{R}\}$ and $\{\eta(t), t \in \mathbb{R}\}$ with zero first moments $E\xi(t) = 0$, $E\eta(t) = 0$, and correlation functions of the form $R_\xi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\lambda}f(\lambda)d\lambda$, $R_\eta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\lambda}g(\lambda)d\lambda$, where $f(\lambda)$ and $g(\lambda)$ are spectral densities of the functions $\xi(t)$ and $\eta(t)$, such that the minimality condition holds true

$$\int_{-\infty}^{\infty} \frac{|\gamma(\lambda)|^2}{f(\lambda)+g(\lambda)} d\lambda < \infty, \qquad (1)$$

where $\gamma(\lambda) = \int_0^\infty \alpha(t)e^{it\lambda}dt$ is nontrivial function of exponential type. Under this condition the error-free extrapolation is impossible [32].

Stationary stochastic processes $\xi(t)$ and $\eta(t)$ admit the spectral decomposition [12]

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} Z_\xi(d\lambda), \eta(t) = \int_{-\infty}^{\infty} e^{it\lambda} Z_\eta(d\lambda), \qquad (2)$$

where $Z_\xi(d\lambda)$ and $Z_\eta(d\lambda)$ are the orthogonal stochastic measures.

Consider the problem of the mean-square optimal linear extrapolation of the functional $A\xi = \int_0^\infty a(t)\xi(t)dt$, which depends on the unknown values of the process $\xi(t)$, based on the observed values of the process $\xi(t) + \eta(t)$ at time points $t \in \mathbb{R}^-\backslash S$, where $S = \bigcup l = 1^s[-M_l - N_l, \ldots, -M_l]$.

Let the function $a(t)$ which defines the functional $A\xi$ satisfy conditions (3)

$$\int_0^\infty |a(t)|\, dt < \infty, \int_0^\infty t\, |a(t)|^2\, dt < \infty. \qquad (3)$$

It follows from the spectral decomposition of the process $\xi(t)$ that the functional $A\xi$ can be represented in the form

$$A\xi = \int_{-\infty}^{\infty} A(e^{i\lambda}) Z_\xi(d\lambda),$$

$$A(e^{i\lambda}) = \int_0^\infty a(t)e^{it\lambda}dt.$$

Denote by $\hat{A}\xi$ the optimal linear estimate of the functional $A\xi$ from the observations of the process $\xi(t)+\eta(t)$. Let $\Delta(f, g) = E\left|A\xi - \hat{A}\xi\right|^2$ be the mean-square error of the estimate $\hat{A}\xi$.

Consider the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ generated by random variables ξ with 0 mathematical expectations, $E\xi = 0$, finite variations, $E|\xi|^2 < \infty$, and inner product $(\xi, \eta) = E\xi\overline{\eta}$. Denote by $H^s(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi(t) + \eta(t) : t \in \mathbb{R}^-\backslash S\}$ in the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$. Let $L_2(f+g)$ be the Hilbert space of complex-valued functions that are square-integrable with respect to the measure whose density is $f(\lambda) + g(\lambda)$, and $L_2^s(f + g)$

be the subspace of $L_2(f+g)$ generated by functions $\{e^{it\lambda}, t \in \mathbb{R}^- \backslash S\}$.

We seek the mean-square optimal linear estimate $\hat{A}\xi$ of the functional $A\xi$ in the form

$$\hat{A}\xi = \int_{-\infty}^{\infty} h(e^{i\lambda})(Z_\xi(d\lambda) + Z_\eta(d\lambda)),$$

where $h(e^{i\lambda}) \in L_2^s(f+g)$ is called spectral characteristic of the estimate.

The mean-square error $\Delta(h;f)$ of the estimate $\hat{A}\xi$ can be calculated by the formula

$$\Delta(h;f,g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|A(e^{i\lambda}) - h(e^{i\lambda})\right|^2 f(\lambda)d\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|h(e^{i\lambda})\right|^2 g(\lambda)d\lambda.$$

According to the Hilbert space projection method proposed by A. N. Kolmogorov [15], the optimal linear estimation of the functional $A\xi$ is a projection of the element $A\xi$ of the space H on the space $H^s(\xi+\eta)$, which can be found from the following conditions:

$$1)\hat{A}\xi \in H^s(\xi+\eta),$$
$$2)A\xi - \hat{A}\xi \perp H^s(\xi+\eta).$$

Therefore, the spectral characteristic $h(e^{i\lambda})$ and the mean-square error $\Delta(h;f,g)$ of the estimate $\hat{A}\xi$ can be calculated by the formulas

$$h(e^{i\lambda}) = A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{C(e^{i\lambda})}{f(\lambda)+g(\lambda)}, \tag{4}$$

$$C(e^{i\lambda}) = \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} (\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt + \int_0^{\infty} (\mathbf{B}^{-1}\mathbf{Ra})(t)e^{it\lambda}dt,$$

$$\begin{aligned} &\Delta(h;f,g) = \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left|A(e^{i\lambda})g(\lambda) + C(e^{i\lambda})\right|^2}{(f(\lambda)+g(\lambda))^2} f(\lambda)d\lambda \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left|A(e^{i\lambda})f(\lambda) - C(e^{i\lambda})\right|^2}{(f(\lambda)+g(\lambda))^2} g(\lambda)d\lambda \\ &= \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle, \end{aligned} \tag{5}$$

where $< A, C >= \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} A(t)\overline{C(t)}dt + \int_0^{\infty} A(t)\overline{C(t)}dt$ is the inner product in the space $L_2(T)$, $(\mathbf{Bx})(t)$, $(\mathbf{Rx})(t)$ and $(\mathbf{Qx})(t)$ are linear operators in the space $L_2(T)$,

$$\begin{aligned} &(\mathbf{Bx})(t) = \\ &\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{1}{f(\lambda)+g(\lambda)}d\lambda du \\ &+\frac{1}{2\pi}\int_0^{\infty} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{1}{f(\lambda)+g(\lambda)}d\lambda du, \end{aligned}$$

$$\begin{aligned} &(\mathbf{Rx})(t) = \\ &\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}d\lambda du \\ &+\frac{1}{2\pi}\int_0^{\infty} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}d\lambda du, \end{aligned}$$

$$\begin{aligned} &(\mathbf{Qx})(t) = \\ &\frac{1}{2\pi}\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}d\lambda du \\ &+\frac{1}{2\pi}\int_0^{\infty} \mathbf{x}(u)\int_{-\infty}^{\infty} e^{i\lambda(u-t)}\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}d\lambda du, \\ &\mathbf{x}(t) \in L_2(T), \quad t \in T, \end{aligned}$$

and the function $\mathbf{a}(t)$ is such that $\mathbf{a}(t) = 0$, $t \in S$, and $\mathbf{a}(t) = a(t)$, $t \geq 0$.

The following theorem holds true.

Theorem 1. *Let the processes $\xi(t)$ and $\eta(t)$ be uncorrelated stationary processes with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean-square error $\Delta(f,g)$ of the optimal linear estimate of the functional $A\xi$ which depends on the unknown values of the process $\xi(j)$ based on observations of the process $\xi(t)+\eta(t)$, $t \in \mathbb{R}^- \backslash S$ can be calculated by formulas (4), (5).*

Consider the problem of the mean-square optimal linear extrapolation of the functional

$$A_N\xi = \int_0^N a(t)\xi(t)dt,$$

which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)+\eta(t)$ at time points $t\in\mathbb{R}^-\backslash S$.

The linear estimate $\hat{A}_N\xi$ of the functional $A_N\xi$ is of the form

$$\hat{A}_N\xi=\int_{-\infty}^{\infty}h_N(e^{i\lambda})(Z_\xi(d\lambda)+Z_\eta(d\lambda)),$$

where $h_N(e^{i\lambda})\in L_2^s(f+g)$ is the spectral characteristic.

Consider the function $\mathbf{a}_N(t)$ such that $\mathbf{a}_N(t)=a(t),t\in S$, $\mathbf{a}_N(t)=a(t),t\in T\cap[0,N]$, $\mathbf{a}_N(t)=0,t\in T\backslash[0,N]$.

Then the spectral characteristic $h_N(e^{i\lambda})$ of the estimate $\hat{A}_N\xi$ can be calculated by the formula

$$h_N(e^{i\lambda})=A_N(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{C_N(e^{i\lambda})}{f(\lambda)+g(\lambda)},\quad(6)$$

$$C_N(e^{i\lambda})=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{R}\mathbf{a}_N)(t)e^{it\lambda}dt+\int_0^{\infty}(\mathbf{B}^{-1}\mathbf{R}\mathbf{a}_N)(t)e^{it\lambda}dt,$$

where $A_N(e^{i\lambda})=\int_0^N a(t)e^{-it\lambda}dt$.

The mean-square error $\Delta(h_N;f,g)$ of the estimate $\hat{A}_N\xi$ can be calculated by the formula

$$\begin{gathered}\Delta(h_N;f,g)=E\left|A_N\xi-\hat{A}_N\xi\right|^2=\\ \langle\mathbf{R}\vec{\mathbf{a}_N},\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}_N}\rangle+\langle\mathbf{Q}\vec{\mathbf{a}_N},\vec{\mathbf{a}_N}\rangle.\end{gathered}\quad(7)$$

We obtain the following corollary.

Corollary 1. *Let the processes $\xi(t)$ and $\eta(t)$ be uncorrelated stationary processes with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h_N(e^{i\lambda})$ and the mean-square error $\Delta(h_N;f,g)$ of the optimal linear estimate of the functional $A_N\xi$ which depends on the unknown values of the process $\xi(j)$ based on observations of the process $\xi(t)+\eta(t)$, $t\in\mathbb{R}^-\backslash S$ can be calculated by formulas (6), (7).*

Consider the case when the stationary process $\xi(t)$ is observed without noise. Then the spectral characteristic of the estimate $\hat{A}\xi$ is of the form

$$h(e^{i\lambda})=A(e^{i\lambda})-C(e^{i\lambda})f^{-1}(\lambda),\quad(8)$$

$$C(e^{i\lambda})=\sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l}(\mathbf{B}^{-1}\mathbf{a})(t)e^{it\lambda}dt+\int_0^{\infty}(\mathbf{B}^{-1}\mathbf{a})(t)e^{it\lambda}dt.$$

The mean-square error of the estimate of the functional can be calculated by formula

$$\Delta(h;f)=<\mathbf{B}^{-1}\mathbf{a},\mathbf{a}>.\quad(9)$$

The following theorem holds true.

Theorem 2. *Let $\xi(t)$ be a stationary stochastic process with the spectral density $f(\lambda)$, which satisfies the minimality condition $\int_{-\pi}^{\pi}f^{-1}(\lambda)d\lambda<\infty$. The spectral characteristic $h(e^{i\lambda})$ and the mean-square error $\Delta(f,g)$ of the optimal linear estimate $\hat{A}\xi$ of the functional $A\xi$ which depends on the unknown values of the process $\xi(j)$ based on observations of the process $\xi(t)$ at time points $t\in\mathbb{R}^-\backslash S$, where $S=\bigcup_{l=1}^{s}[-M_l-N_l,\dots,-M_l]$, can be calculated by formulas (8),(9).*

Minimax method of extrapolation

The results from the section above can be applied to the solution of the introduced problem only in the case when spectral densities of the processes are exactly known. In the case when the full information on spectral densities is impossible, but it is known that spectral densities belong to the specified class of admissible densities, the minimax approach will be useful. The purpose of this method is to find estimate that minimize the maximum values of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities.

Let us introduce definitions [21].

Definition 1. *For a given class of spectral densities $D=D_f\times D_g$ the spectral densities $f_0(\lambda)\in D_f$, $g_0(\lambda)\in D_g$ are called least favorable in the class D for the optimal linear extrapolation of the functional $A\xi$ if the* *following relation holds true*

$$\Delta(f_0, g_0) = \Delta(h(f_0, g_0); f_0, g_0)$$
$$= \max_{(f,g)\in D_f\times D_g} \Delta(h(f,g); f, g).$$

Definition 2. *For a given class of spectral densities $D = D_f \times D_g$ the spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear extrapolation of the functional $A\xi$ is called minimax-robust if there are satisfied conditions*

$$h^0(e^{i\lambda}) \in H_D = \bigcap_{(f,g)\in D_f\times D_g} L_2^s(f+g),$$

$$\min_{h\in H_D}\max_{(f,g)\in D} \Delta(h; f, g) = \max_{(f,g)\in D} \Delta\left(h^0; f, g\right).$$

Making use of the definitions above and the results from the previous section, we can formulate the following lemmas.

Lemma 1. *Spectral densities $f_0(\lambda) \in D_f$, $g_0(\lambda) \in D_g$ satisfying the minimality condition (1) are the least favorable in the class $D = D_f \times D_g$ for the optimal linear extrapolation of the functional $A\xi$, if the Fourier coefficients of the functions $(f_0(\lambda)+g_0(\lambda))^{-1}$, $f_0(\lambda)(f_0(\lambda)+g_0(\lambda))^{-1}$, $f_0(\lambda)g_0(\lambda)(f_0(\lambda)+g_0(\lambda))^{-1}$ determine the operators $\mathbf{B}^0, \mathbf{R}^0, \mathbf{Q}^0$, which determine a solution to the constrain optimization problem*

$$\max_{(f,g)\in D_f\times D_g} \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle = \langle \mathbf{R}^0\vec{\mathbf{a}}, (\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}^0\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle. \tag{10}$$

The minimax spectral characteristic $h^0 = h(f_0, g_0)$ is determined by the formula (4) if $h(f_0, g_0) \in H_D$.

Corollary 2. *Suppose the spectral density $f_0(\lambda) \in D_f$ is such that $f_0^{-1}(\lambda)$ is integrable. The spectral density $f_0(\lambda) \in D_f$ is the least favorable in the class D_f for the optimal linear extrapolation of the functional $A\xi$ from the observation of the process $\xi(t)$ at time points $t \in \mathbb{R}^-\backslash S$, if the Fourier coefficients of the function $f_0^{-1}(\lambda)$ determine the operator $\mathbf{B}^0$, which determines a solution to the constrain optimization problem*

$$\max_{f\in D_f} \langle \mathbf{B}^{-1}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle = \langle (\mathbf{B}^0)^{-1}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle. \tag{11}$$

The minimax spectral characteristic $h^0 = h(f_0)$ is determined by the formula (8) if $h(f_0) \in H_{D_f}$.

The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ and the minimax spectral characteristic $h^0 = h(f_0, g_0)$ form a saddle point of the function $\Delta(h; f, g)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta\left(h^0; f, g\right) \le \Delta\left(h^0; f_0, g_0\right) \le \Delta(h; f_0, g_0),$$
$$\forall h \in H_D, \forall f \in D_f, \forall g \in D_g,$$

hold true if $h^0 = h(f_0, g_0)$ та $h(f_0, g_0) \in H_D$, where (f_0, g_0) is a solution to the constrained optimization problem

$$\sup_{(f,g)\in D_f\times D_g} \Delta(h(f_0, g_0); f, g) = \Delta(h(f_0, g_0); f_0, g_0), \tag{12}$$

$$\Delta(h(f_0, g_0); f, g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left|A(e^{i\lambda})g_0(\lambda) + C^0(e^{i\lambda})\right|^2}{(f_0(\lambda)+g_0(\lambda))^2} f(\lambda)d\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left|A(e^{i\lambda})f_0(\lambda) - C^0(e^{i\lambda})\right|^2}{(f_0(\lambda)+g_0(\lambda))^2} g(\lambda)d\lambda,$$

$$C^0(e^{i\lambda}) = \sum_{l=1}^{s}\int_{-M_l-N_l}^{-M_l} ((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt + \int_0^{\infty} ((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\lambda}dt, \quad t \in S.$$

The constrained optimization problem (12) is equivalent to the unconstrained optimization problem [30]:

$$\Delta_D(f, g) = -\Delta(h(f_0, g_0); f, g) + \delta((f, g)\,|D_f \times D_g) \to \inf, \tag{13}$$

where $\delta((f, g)\,|D_f \times D_g)$ is the indicator function of the set $D = D_f \times D_g$. Solution of the problem (13) is characterized by the condition $0 \in \partial\Delta_D(f_0, g_0)$, where $\partial\Delta_D(f_0)$ is the subdifferential of the convex functional $\Delta_D(f, g)$ at point (f_0, g_0) [31].

The form of the functional $\Delta(h(f_0, g_0); f, g)$ admits finding the derivatives and differentials of the functional in the space $L_1 \times L_1$. Therefore the complexity of the optimization problem (13) is determined by the complexity of calculating the subdifferential of the indicator functions $\delta((f,g)|D_f \times D_g)$ of the sets $D_f \times D_g$ [11].

Lemma 2. *Let (f_0, g_0) be a solution to the optimization problem (13). The spectral densities $f_0(\lambda)$, $g_0(\lambda)$ are the least favorable in the class $D = D_f \times D_g$ and the spectral characteristic $h^0 = h(f_0, g_0)$ is the minimax of the optimal linear estimate of the functional $A\xi$ if $h(f_0, g_0) \in H_D$.*

Least favorable spectral densities in the class $D = D_0 \times D_\varepsilon^1$

Consider the problem of extrapolation of the functional $A\xi$ in the case when spectral densities of the processes belong to the class of admissible spectral densities $D = D_0 \times D_\varepsilon^1$,

$$D_0 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) d\lambda \leq P_1 \right. \right\},$$

$$D_\varepsilon^1 = \left\{ g(\lambda) \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)| \, d\lambda \leq \varepsilon \right. \right\}$$

where spectral density $g_1(\lambda)$ is known and fixed. Class D_ε^1 describes a "ε-district" in the space L_1 of the given bounded spectral density $g_1(\lambda)$.

Consider the spectral densities such that $f_0(\lambda) \in D_0$, $g_0(\lambda) \in D_\varepsilon^1$. Suppose the following functions are bounded

$$h_f(f_0, g_0) = \frac{\left|A(e^{i\lambda}) g_0(\lambda) + C^0(e^{i\lambda})\right|^2}{(f_0(\lambda) + g_0(\lambda))^2}, \quad (14)$$

$$h_g(f_0, g_0) = \frac{\left|A(e^{i\lambda}) f_0(\lambda) - C^0(e^{i\lambda})\right|^2}{(f_0(\lambda) + g_0(\lambda))^2}. \quad (15)$$

Then the functional $\Delta(h(f_0, g_0); f, g)$ is continuous and bounded in the space $L_1 \times L_1$

$$\Delta(h(f_0, g_0); f, g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_f(f_0, g_0) f(\lambda) d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} h_g(f_0, g_0) g(\lambda) d\lambda.$$

Hence, the following relation holds true [30]

$$\partial \Delta_{D_0 \times D_\varepsilon^1}(f_0, g_0) = -\partial \Delta(h(f_0, g_0); f_0, g_0) + \partial \delta((f_0, g_0) \left| D_0 \times D_\varepsilon^1 \right.).$$

Condition $0 \in \partial \Delta_{D_0 \times D_\varepsilon^1}(f_0, g_0)$ makes it possible to find equations which the least favorable densities satisfy

$$\left|A(e^{i\lambda}) g_0(\lambda) + C^0(e^{i\lambda})\right| = \alpha_1 (f_0(\lambda) + g_0(\lambda)), \quad (16)$$

$$\left|A(e^{i\lambda}) f_0(\lambda) - C^0(e^{i\lambda})\right| = (f_0(\lambda) + g_0(\lambda)) \Psi(\lambda) \alpha_2, \quad (17)$$

where $|\Psi(\lambda)| \leq 1$ and $\Psi(\lambda) = sign(g_0(\lambda) - g_1(\lambda))$, when $g_0(\lambda) \neq g_1(\lambda)$, constants $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. Particularly, $\alpha_1 \neq 0$, if $\frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(\lambda) d\lambda = P_1$.

Equations (16), (17) together with the optimization problem (10) and normality condition

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)| \, d\lambda = \varepsilon \quad (18)$$

determine the least favorable spectral densities in the class D.

Theorem 3. *Let the spectral densities $f_0(\lambda) \in D_0$, $g_0(\lambda) \in D_\varepsilon^1$ satisfy the minimality condition (1), and functions determined by formulas (14), (15) be bounded. Spectral densities $f_0(\lambda)$, $g_0(\lambda)$ are the least favorable in the class $D_0 \times D_\varepsilon^1$ for the optimal linear extrapolation of the functional $A\xi$ if they satisfy equations (16)–(18) and determine a solution to the optimization problem (10). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ is determined by formula (4).*

Theorem 4. *Suppose that $f_0(\lambda) \in D_{\varepsilon_1}^1$, $g_0(\lambda) \in D_{\varepsilon_2}^1$, where*

$$D_{\varepsilon_1}^1 = \left\{ f(\lambda) \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda) - f_1(\lambda)| \, d\lambda \leq \varepsilon_1 \right. \right\},$$

$$D_{\varepsilon_2}^1 = \left\{ g(\lambda) \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)| \, d\lambda \leq \varepsilon_2 \right. \right\},$$

spectral densities $f_1(\lambda)$, $g_1(\lambda)$ are known and fixed. Let the spectral densities $f_0(\lambda)$, $g_0(\lambda)$ satisfy the minimality condition (1) and functions determined by formulas (14), (15) be bounded. Spectral densities $f_0(\lambda)$, $g_0(\lambda)$ are the least favorable in the class $D_{\varepsilon_1}^1 \times D_{\varepsilon_2}^1$ for the optimal linear extrapolation of the functional $A\xi$ if they satisfy equations

$$\left|A(e^{i\lambda}) g_0(\lambda) + C^0(e^{i\lambda})\right| = (f_0(\lambda) + g_0(\lambda)) \Psi_1(\lambda) \alpha_1,$$

$$\left|A(e^{i\lambda})f_0(\lambda)-C^0(e^{i\lambda})\right|=(f_0(\lambda)+g_0(\lambda))\Psi_2(\lambda)\alpha_2,$$

where $|\Psi_1(\lambda)|\leq 1$ *and* $\Psi_1(\lambda)=sign(f_0(\lambda)-f_1(\lambda))$, *when* $f_0(\lambda)\neq f_1(\lambda)$, $|\Psi_2(\lambda)|\leq 1$ *and* $\Psi_2(\lambda)=sign(g_0(\lambda)-g_1(\lambda))$, *when* $g_0(\lambda)\neq g_1(\lambda)$, *constants* $\alpha_1\geq 0$, $\alpha_2\geq 0$.

A pair $(f_0(\lambda),g(\lambda))$ *of the least favorable densities determines a solution to the optimization problem (10) and satisfies conditions*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(\lambda)-f_1(\lambda)|\,d\lambda=\varepsilon_1,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|g(\lambda)-g_1(\lambda)|\,d\lambda=\varepsilon_2.$$

The function calculated by the formula (4) is the minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$.

Corollary 3. *Suppose the spectral density* $g(\lambda)$ *is known, the spectral density* $f_0(\lambda)\in D^1_{\varepsilon_1}$. *Let the function* $f_0(\lambda)+g(\lambda)$ *satisfy the minimality condition (1), the function* $h_f(f_0,g)$ *determined by formula (14) be bounded. The spectral density* $f_0(\lambda)$ *is the least favorable in the class* $D^1_{\varepsilon_1}$ *for the optimal linear extrapolation of the functional* $A\xi$ *if it is of the form* $f_0(\lambda)=\max\left\{f_1(\lambda),\alpha_1\left|A(e^{i\lambda})g(\lambda)+C^0(e^{i\lambda})\right|-g(\lambda)\right\}$, *and the pair* $(f_0(\lambda),g(\lambda))$ *is a solution of the optimization problem (10). The minimax-robust spectral characteristic of the optimal estimate of the functional* $A\xi$. *is determined by formula (4).*

Corollary 4. *Let the spectral density* $f_0(\lambda)\in D^1_{\varepsilon_1}$, *the function* $f_0^{-1}(\lambda)$ *is integrable and the function determined by the formula (8) be bounded. The spectral density* $f_0(\lambda)$ *is the least favorable in the class* $D^1_{\varepsilon_1}$ *for the optimal linear extrapolation of the functional* $A\xi$ *if it satisfies the following relation* $\left|C^0(e^{i\lambda})\right|=f_0(\lambda)\Psi_1(\lambda)\alpha_1$, *and* $f_0(\lambda)$ *determines a solution to the optimization problem (11). The function calculated by the formula (8) is the minimax-robust spectral characteristic of the optimal estimate of the functional* $A\xi$.

Conclusions

In the article we propose methods of the mean-square optimal linear extrapolation of the functional which depends on the unknown values of stationary stochastic process based on observed data of the process with noise and missing values. In the case of spectral certainty when the spectral densities of the stationary processes are known we derive formulas for calculating the spectral density and the mean-square error of the estimate of the functional. Results of solution of the estimation problem are obtained for the case of observations without noise. In the case of spectral uncertainty, when specified sets of admissible densities are given, we derive the equations which determine the least spectral densities.

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